MATH2050B 2021 HW 3

TA's solutions[1] to selected problems

Q1. Solve the inequality system

$$4 < |x+2| + |x-1| \le 5.$$

Solution. We divide the real line into three parts to see if there is any solution in each part:

- **Case 1:** $x \le -2$, then $|x+2|+|x-1| = -2-x+1-x = -1-2x$. In this case x satisfies the inequality iff $x \in (-3, -5/2]$.
- **Case 2:** $-2 < x \le 1$, then $|x+2|+|x-1| = 3$. In this case there is no x satisfies the inequality.
- **Case 3:** $1 < x$, then $|x+2|+|x-1| = 2x+1$. In this case x satisfies the inequality iff $x \in (3/2, 2]$.

Thus the solution set A is $[-3, -5/2) \cup (3/2, 2]$.

Q2. Show by MI the binomial formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{n-k} a^{n-k} b^k.$$

And for $a > 0$, show that

$$(1+a)^n \ge 1 + na$$

and

$$(1+a)^n \ge \frac{n(n-1)(n-2)}{3!} a^2, \qquad \forall n \ge 3$$

so

$$\frac{n^2}{(1+a)^n} \to 0 \text{ and similarly } \frac{n^{10000}}{1.000001^n} \to 0$$

Solution. When $n = 0, 1$, the binomial formula is clearly true. Suppose that the binomial formula is true for $n = 0, 1, 2, \ldots, N$. Now

$$\begin{aligned}
(a+b)^{N+1} &= (a+b) \sum_{k=0}^{N} \binom{N}{N-k} a^{N-k} b^k \\
&= \sum_{k=0}^{N} \binom{N}{N-k} a^{N+1-k} b^k + \sum_{k=0}^{N} \binom{N}{N-k} a^{N-k} b^{k+1} \\
&= \binom{N}{N} a^{N+1} + \sum_{k=1}^{N} \binom{N}{N-k} a^{N+1-k} b^k + \sum_{k=0}^{N-1} \binom{N}{N-k} a^{N-k} b^{k+1} + \binom{N}{0} b^{N+1} \\
&= a^{N+1} + \sum_{k=0}^{N-1} \binom{N}{N-k-1} a^{N-k} b^{k+1} + \sum_{k=0}^{N-1} \binom{N}{N-k} a^{N-k} b^{k+1} + b^{N+1}
\end{aligned}$$

[1] please kindly send an email to `nclliu@math.cuhk.edu.hk` if you have spotted any typo/error/mistake.

$$
\begin{aligned}
&= a^{N+1} + \sum_{k=0}^{N-1} [\binom{N}{N-k-1} + \binom{N}{N-k}] a^{N-k} b^{k+1} + b^{N+1} \\
&= a^{N+1} + \sum_{k=0}^{N-1} \binom{N+1}{N-k} a^{N-k} b^{k+1} + b^{N+1} \\
&= a^{N+1} + \sum_{k=1}^{N} \binom{N+1}{N+1-k} a^{N+1-k} b^{k} + b^{N+1} \\
&= \sum_{k=0}^{N+1} \binom{N+1}{N+1-k} a^{N+1-k} b^{k}.
\end{aligned}
$$

Thus the binomial formula also holds for $n = N+1$. By MI, the binomial formula is true for all n.

To show $(1+a)^n \geq 1 + na$, note $1, na$ are the first two terms of the binomial expansion

$$(1+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k.$$

Since all terms are non-negative, so $(1+a)^n \geq 1 + na$.

The next ineq to be proved is $(1+a)^n \geq a^2 n(n-1)(n-2)/3!$. The inequality should be

$$(1+a)^n \geq \frac{n(n-1)(n-2)}{3!} a^3,$$

which can also be obtained from the binomial theorem.

(Fact: $(1+a)^n \geq a^2 n(n-1)(n-2)/3!$ holds iff $n < 13$)

Q3. Let $x_1 > 0$, and $x_{n+1} = x_n + \frac{1}{x_n}$. Show that

(i) $x_{n+1} \geq x_n$ and $x_{n+1}^2 \geq x_{n+1} x_n = x_n^2 + 1$

(ii) (x_n^2) and (x_n) are unbounded.

(iii) Can (x_n) converge? Hint: either use the necessary condition theorem or by contradiction argument.

Solution. (i) Note that $x_n > 0$ for all n, so $x_{n+1} = x_n + \frac{1}{x_n} \geq x_n$ and $x_{n+1}^2 \geq x_{n+1} x_n$.

(ii) It suffices to prove that (x_n) is unbounded, because if (x_n) is unbounded then (x_n^2) must be unbounded.(Why?) Suppose that (x_n) were bounded, then by the Monotone Convergence Theorem for sequences, (x_n) is convergent. Notice that $x = \lim_{n\to\infty} x_n = \sup_n x_n \geq x_1 > 0$ and $x_{n+1} = x_n + \frac{1}{x_n}$, so

$$x = x + \frac{1}{x}$$

Then a contradiction arises because $0 = \frac{1}{x}$. Hence (x_n) is unbounded.

(iii) follows from (ii).

Q4. Let $x_{n+1} = 2 + \frac{x_n}{2}$ for $n > 1$. In each of the following cases, is (x_n) monotone/convergent?

(i) $x_1 = 1$

(ii) $x_1 = 10$

Determine the limit when exists.

Solution. (i) The sequence is increasing and is bounded above by 4.

One computes $x_2 = \frac{5}{2}$. It is clear that $x_1 < x_2 < 4$. Suppose that for some k, we have

$$x_1 < x_2 < \cdots < x_k < 4$$

then $x_{k+1} = 2 + \frac{x_n}{2} < 2 + \frac{4}{2} = 4$ and notice that

$$x_{k+1} - x_k = 2 - \frac{x_k}{2} > 2 - \frac{4}{2} = 0$$

Therefore by induction, (x_n) is increasing and is bounded above by 4. Since $x_{n+1} = 2 + \frac{x_n}{2}$, we find out that the limit is 4.

(ii) is similar to (i) (the details are skipped here), the sequence is decreasing and is bounded below by 4. The limit is 4.